

This question paper contains 4 printed pages.

Your Roll No.

S. No. of Paper : 93
Unique Paper Code : 32351501
Name of the Paper : Metric Spaces
Name of the Course : B.Sc. (Hons.) Mathematics
Semester : V
Duration : 3 hours
Maximum Marks : 75

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(Write your Roll No. on the top immediately
on receipt of this question paper.)

Attempt any two parts from each question.
All questions are compulsory

1. (a) (i) Let $X = \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$. Define the metric d on X by :

$$d(x, y) = \tan^{-1} |x - \tan^{-1} y|, x, y \in X,$$
where $\tan^{-1}(\infty) = \pi/2$ and $\tan^{-1}(-\infty) = -\pi/2$.
Show that (X, d) is a metric space.

(ii) Let X denote the set of all Riemann integrable functions on $[a, b]$. For f, g in X , define:

$$d(f, g) = \int_a^b |f(x) - g(x)| dx.$$

Show that d is not a metric on X . 3+3=6

(b) Prove that a sequence in \mathbf{R}^n is Cauchy in the Euclidean metric d_2 if and only if it is Cauchy in the maximum metric d_∞ . 6

P. T. O.

(c) (i) Show that the metric space (X, d) of rational numbers is an incomplete metric space.

(ii) Let X be any nonempty set and d be the discrete metric defined on X . Prove that the metric space (X, d) is a complete metric space. $3+3=6$

2. (a) Let (X, d) be a metric space. Prove that the intersection of any finite family of open sets in X is an open set in X . Is it true for the intersection of an arbitrary family of open sets? Justify your answer.

(b) Prove that if A is a subset of the metric space (X, d) , then $d(A) = d(\bar{A})$.

(c) Let F be a subset of a metric space (X, d) . Prove that the following are equivalent:

(i) $x \in \bar{F}$

(ii) $S(x, \epsilon) \cap F \neq \emptyset$ for every open ball $S(x, \epsilon)$ centered at x ;

(iii) There exists an infinite sequence $\{x_n\}$, $n \geq 1$ of points (not necessarily distinct) of F such that $x_n \rightarrow x$.

3. (a) Let (X, d) be a metric space and $Z \subseteq Y \subseteq X$. If $cl_X(Z)$ and $cl_Y(Z)$ denote, respectively, the closures of Z in metric spaces X and Y , then show that:

$$cl_Y(Z) = Y \cap cl_X(Z).$$

(b) (i) Let Y be a nonempty subset of a metric space (X, d_X) , and (Y, d_Y) is complete. Show that Y is closed in X .

(ii) Is the converse of part (i) true? Justify your answer. 4+2=6

(c) Let d_p ($p \geq 1$) on the set \mathbb{R}^n be given by:

$$d_p(x, y) = \left(\sum_{j=1}^n |x_j - y_j|^p \right)^{1/p},$$

for all $x=(x_1, x_2, \dots, x_n), y=(y_1, y_2, \dots, y_n)$ in \mathbb{R}^n . Show that (\mathbb{R}^n, d_p) is a separable metric space. 6

4. (a) Prove that a mapping $f: (X, d_X) \rightarrow (Y, d_Y)$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y . 6 ½

(b) (i) Define an isometry between the metric spaces (X, d_X) and (Y, d_Y) , and show that it is a homeomorphism.

(ii) Is the completeness of a metric space preserved under homeomorphism? Justify your answer. 4+2½=6½

(c) State and prove the Contraction Mapping Principle. 1½+5=6½

5. (a) Let f be a mapping of (X, d_X) into (Y, d_Y) . Prove that f is continuous on X if and only if for every subset F of Y :

$$f^{-1}(F^o) \subseteq (f^{-1}(F))^o \quad 6\frac{1}{2}$$

P. T. O.

- (b) Prove that the metrics d_1 , d_2 and d_∞ defined on \mathbb{R}^n by:

$$d_1(x, y) = \sum_{j=1}^n |x_j - y_j|;$$

$$d_2(x, y) = \left(\sum_{j=1}^n |x_j - y_j|^2 \right)^{1/2}; \text{ and}$$

$$d_\infty(x, y) = \max \{ |x_j - y_j| : j = 1, 2, \dots, n \}$$

for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

are equivalent.

- (c) Prove that a metric space (X, d) is disconnected if and only if there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0) . $6\frac{1}{2}$
6. (a) If every two points in a metric space X are contained in some connected subset of X , prove that X is connected. $6\frac{1}{2}$
- (b) Let (X, d) be a metric space and Y a subset of X . Prove that if Y is a compact subset of (X, d) , then Y is bounded. Is the converse true? Justify your answer. $6\frac{1}{2}$
- (c) If f is a one-to-one continuous mapping of a compact metric space (X, d_X) onto a metric space (Y, d_Y) , then prove that f is a homeomorphism. $6\frac{1}{2}$